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**WP MRE 2019.3**

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# On Two Dominances of Fuzzy Variables based on a Parametric Fuzzy Measure and Application to Portfolio Selection with Fuzzy Return

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April 16, 2018

**Abstract**

Iwamura [18] introduced a new parametric fuzzy measure as a convex linear combination of possibility and necessity measures. This measure generalizes the credibility measure and the parameter of the possibility measure is considered as the decision making (investors) optimism's level. In this paper, we introduce by means of that measure two new dominances (binary relations) on fuzzy variables. The first one generalizes the first order dominance introduced recently by Tassak et al. [17] and the second one, based on optimism's level and called optimisnism dominance, is stronger than the first one. We study properties of these dominances on trapezoidal fuzzy numbers and we characterize them. We implement the optiminism dominance in a numerical example to display that its set of efficient portfolios enlarges the set of efficient portfolios obtained by Tassak et al. [17] through their first order dominance.

*Keywords:* Fuzzy variable; Parametric fuzzy measure; Generalized First order dominance; Optiminism Dominance; Set of best portfolios

# 1 Introduction

Portfolio selection theory consists on the choice of "best portfolios" among several other ones in the context that assets' future returns are uncertain ([10], [2], [15], [13], [16], [5], [6], [14]). Therefore, the literature presents a huge variety of methods for the determination of efficient portfolios. More precisely, one can subdivide those methods following two approaches: the first one deals with investors' preferences through proposed target values for benefit or investment risk and the second one doesn't take into consideration any target value and thus, proposes various efficient portfolios on which each investor can make a choice based on his preference. However, two mathematical tools are used as supports to analyze those approaches. Moments and semi-moments of a (random or fuzzy) variable, defined by means of crisp or fuzzy measure, are used in optimization models for the first approach ([10],[13], [5], [6], [14],[3]) whereas the core of portfolios or the set of best portfolios based on the dominance relation on (random or fuzzy) variables defined by means of the fuzzy measure are main tools of the second approach ([12], [17]). In this paper, we focus on the second approach, namely Game theoretical approach, by introducing, characterizing and studying properties of the two new dominance relations comparing variables.
However, the literature also presents more than one way to describe uncertainty in various phenomena and some usual concepts related to human being perceptions. As a consequence, in portfolio selection, future asset's return can be described by a random or fuzzy variable according to the investment context. It means that, in some context, only randomness can influence future assets' returns and leads to random variable whereas in other context, uncertainty is due to imprecise or vague concepts used to describe future assets' returns (around 3UM, between 4UM and 5UM, very expensive...) and leads to a fuzzy variable. This work deals with fuzzy variable which describes uncertainty using the generalized fuzzy measure. Notice that this recent measure, defined by Iwamura [18] as a convex combination of possibility and necessity measures defined by Zadeh [20], takes into consideration an investor optimism's level towards risk ([18], [3]). This paper is a modest contribution for the study of dominance relations on fuzzy variables using the new generalized measure and the application of obtained results to determine best portfolios made up of assets with fuzzy return. More precisely, we define a distribution function of a fuzzy variable by means of that measure and we study its properties. This distribution is used to define two new dominance relations that allow to compare fuzzy variables according to two main assumptions: investors have the same optimism's level towards investment risk or not. Then, these dominance relations are characterized on trapezoidal fuzzy variables and other properties are established. However, since it seems more natural to compare two entities before selecting them,

binary comparisons of fuzzy variables (through distribution functions) can lead to portfolio selection. We prove in particular that the second dominance relation is a pre-order and it is complete in a particular subset of trapezoidal fuzzy variables. This "local completeness" is a specificity of the second order dominance in that sense it can allow in some situations to classify fuzzy variables and in this particular context, it allows to an investor to classify different assets of his portfolio, which naturally makes easier the selection. Furthermore, we describe the sets of non dominated portfolios according to the two dominances and we establish a link between the two sets. We use the seven assets proposed by Huang [5] to implement the second dominance in order to obtain non dominated portfolios. Those obtained portfolios complement the set of best portfolios obtained by first authors ([5], [6], [14], [17]).
The paper is organized as follows. Section 2 gives some preliminaries on fuzzy subsets and on fuzzy measures. It presents the new distribution functions of a fuzzy variable and links with known distribution functions. Section 3 presents the new dominance relations, their characterizations and their properties. In Section 4, the new dominance relation is implemented on a set of seven fuzzy variables and some non dominated portfolios are proposed. Section 5 gives concluding remarks and the Appendix made up of some proofs.

## 2 Preliminaries

Throughout this paper, $X$ is a nonempty set namely the universal set.

### 2.1 Fuzzy subsets and fuzzy measures

A fuzzy subset $A$ of $X$ is defined by its membership function: $\mu_A : X \to [0,1]$ such that, to each $x \in X$, is associated $\mu_A(x)$. For $x \in X$, $\mu_A(x)$ represents the membership grade of $x$ to $A$. If $\forall x \in A, \mu_A(x) \in \{0,1\}$, then $A$ becomes a crisp subset of $X$. The support of $A$ is the crisp subset denoted by $Supp(A)$ and defined by: $Supp(A) = \overline{\{x \in X, \mu_A(x) > 0\}}$.

Let us recall the parametric measure $M_\lambda$ defined by Yang and Iwamura [18] that we will use in this paper.
For $A \subset X$ and $\xi : X \to \mathbb{R}$ a fuzzy variable with membership function $\mu$, that means, each element $x$ of $X$ is associated to a real number $\xi(x)$ by means of $\xi$ with the membership grade $\mu(\xi(x))$, $M_\lambda(A) = \lambda Pos(A) + (1-\lambda)$ Nec(A) with $\lambda \in [0,1]$. We recall that if $\lambda = 1$ (respectively $\lambda = 0$), then $M_\lambda$ becomes possibility measure defined as $Pos(A) = \sup_{x\in A} \mu(\xi(x))$ (respectively necessity measure defined as $Nec(A) = 1 - Pos(A^c) = \inf_{x\in A} \mu(\xi(x))$) by Zadeh [20]. If $\lambda = \frac{1}{2}$, then $M_\lambda$ becomes credibility measure defined as $Cred(A) = \frac{1}{2}[Pos(A) + Nec(A)]$ by Liu [7].

Denote that $\lambda$ is the optimistic level of an investor or a decision maker. Thus, $M_{\lambda_1}$ and $M_{\lambda_2}$ can be considered as two different ways to measure fuzzy event $A$, depending of two investors with optimistic levels $\lambda_1$ and $\lambda_2$, by the combination of possibility and necessity measures.

In the next Subsection, we define and study the distribution function of a fuzzy variable with respect to $M_\lambda$-measure.

## 2.2 Distribution function of a fuzzy variable with respect to $M_\lambda$-measure

**Definition 1.** *Let $\xi$ be a fuzzy variable and $\lambda \in [0,1]$. The distribution function $\phi_\lambda$ of $\xi$ with respect to $M_\lambda$-measure is given by:* $\forall x \in \mathbb{R}$,

$$\phi_{\lambda,\xi}(x) = M_\lambda\{\xi \leq x\}.$$

If there is no confusion, $\phi_{\lambda,\xi}$ will be simply denoted $\phi_\lambda$. In addition, if $\lambda = \frac{1}{2}$, that is, $M_\lambda$ is the credibility measure and $\phi_{\lambda,\xi}$ is simply denoted $\phi_\xi$ or simply $\phi$ defined by Liu [9].

Let us define the distribution function of a trapezoidal fuzzy variable.

**Example 1.** *Let $\xi = (a,b,c,d)$ be a trapezoidal fuzzy variable, $\lambda \in [0,1]$ and $x \in \mathbb{R}$. We have:*

$$\phi_\lambda(x) = \begin{cases} 1 \ \textit{if}\ d \leq x \\ \frac{\lambda(d-x)+x-c}{d-c} \ \textit{if}\ c \leq x \leq d \\ \lambda \ \textit{if}\ b \leq x \leq c \\ \frac{\lambda(x-a)}{b-a} \ \textit{if}\ a \leq x \leq b \\ 0 \ \textit{if}\ x \leq a \end{cases}. \quad (1)$$

Let us end this Subsection by establishing the two next results on properties of $\phi_\lambda$.

The first result shows that the cumulative function of a fuzzy variable is not decreasing w.r.t. the optimistic level.

**Proposition 1.** *Let $\xi$ be a fuzzy variable. The distribution function $\phi_\lambda$ of $\xi$ is nondecreasing with respect to $\lambda$.*

**Proof**: This result is justified by the fact that the $M_\lambda$-measure is nondecreasing with respect to $\lambda$.
In fact, by setting $f(\lambda) = \lambda Pos(A) + (1-\lambda)Nec(A)$ with $\lambda \in [0,1]$ and for a fixed subset $A$ of $\mathbb{R}$, we have: $f'(\lambda) = Pos(A) - Nec(A) \geq 0$ as $Pos(A) \geq Nec(A)$. $\square$

The second result establishes that in the set of trapezoidal fuzzy variables, if two cumulative functions coincide for two optimistic levels, then they will coincide as credibilistic cumulative functions. The proof of this result is given in Appendix.

**Proposition 2.** *Let $\xi_1 = (a_1, b_1, c_1, d_1)$ and $\xi_2 = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy variables whose respective distribution functions are $\phi_\lambda^1$ and $\phi_\lambda^2$ with respect to $M_\lambda$-measure.*
*If there exists $\lambda_1, \lambda_2 \in [0;1]$ such that: $\phi_{\lambda_1}^1(x) = \phi_{\lambda_2}^2(x), \forall x \in \mathbb{R}$, then*

$$\phi_1(x) = \phi_2(x), \forall x \in \mathbb{R}.$$

In the next Section, we introduce and study two new order dominances of fuzzy variables which take into account investors's optimism level $\lambda$. Therefore, the first one compares fuzzy variables with the same optimism level from investors whereas the second one compares them with different optimism levels from investors.

# 3 Dominance relations of fuzzy variables with respect to $M_\lambda$-measure

## 3.1 First order dominance: Definition and Properties

**Definition 2.** *Let $\lambda \in [0;1]$ and $\xi_1$, $\xi_2$ be two fuzzy variables whose respective distribution functions are $\phi_\lambda^1$ and $\phi_\lambda^2$ with respect to $M_\lambda$-measure.*
*The first order dominance is the binary relation on fuzzy variables denoted by $\succeq_{d_1^\lambda}$ and defined by: $\xi_1 \succeq_{d_1^\lambda} \xi_2$ if $\forall x \in \mathbb{R}, \phi_\lambda^1(x) \leq \phi_\lambda^2(x)$.*

We establish four first properties of the dominance $\succeq_{d_1^\lambda}$ for trapezoidal fuzzy variables. The three first ones shows that $\succeq_{d_1^\lambda}$ is a pre-order on the set $\mathcal{A}$ of trapezoidal fuzzy variables whereas the last one establishes sufficient condition for comparison on $\mathcal{A}$.

**Proposition 3.** *Let $\lambda \in [0;1]$.*

1. *Reflexivity: $\forall \xi \in \mathcal{A}$, $\xi \succeq_{d_1^\lambda} \xi$;*
2. *Symmetry: $\forall (\xi, \eta) \in \mathcal{A}^2$, if $\xi \succeq_{d_1^\lambda} \eta$ and $\eta \succeq_{d_1^\lambda} \xi$ then $\xi \sim \eta$;*
3. *Transitivity: $\forall (\xi, \eta, \theta) \in \mathcal{A}^3$, if $\xi \succeq_{d_1^\lambda} \eta$ and $\eta \succeq_{d_1^\lambda} \theta$ then $\xi \succeq_{d_1^\lambda} \theta$;*
4. *$\forall (\xi, \eta) \in \mathcal{A}^2$, $\inf Supp(\xi) > \sup Supp(\eta) \Rightarrow \xi \succeq_{d_1^\lambda} \eta$.*

**Proof**: Proofs are similar to ones established by Tassak et al. [17] for the first order dominance with respect to credibility measure. □

The following result characterizes $\succeq_{d_1^\lambda}$ for trapezoidal fuzzy variables.

**Theorem 1.** *Let* $\xi_1 = (a_1, b_1, c_1, d_1)$ *and* $\xi_2 = (a_2, b_2, c_2, d_2)$ *be two trapezoidal fuzzy variables and* $\lambda \in [0, 1]$. *Then,*

$$\xi_1 \succeq_{d_1^\lambda} \xi_2 \Leftrightarrow \begin{cases} a_1 \geq a_2 \\ b_1 \geq b_2 \\ c_1 \geq c_2 \\ d_1 \geq d_2 \end{cases}.$$

**Proof**: This proof is similar to the proof of the characterization of first order dominance with respect to credibility measure proposed by Tassak et al. [17]. □

Let us notice that according to Eq. 1, all dominances $\succeq_{d_1^\lambda}$ with $\lambda \in [0, 1]$ are equivalent on $\mathcal{A}$. But, it is not longer the case for the comparison of two fuzzy variables with different membership functions' sharps.

The following result establishes the compatibility of $\succeq_{d_1^\lambda}$ with respect to the sum.

**Proposition 4.** *Let* $\xi_1 = (a_1, b_1, c_1, d_1)$, $\xi_2 = (a_2, b_2, c_2, d_2)$ *and* $\xi_3 = (a_3, b_3, c_3, d_3)$ *be three trapezoidal fuzzy variables and* $\lambda \in [0, 1]$.

$$\xi_1 \succeq_{d_1^\lambda} \xi_2 \Rightarrow \xi_1 + \xi_3 \succeq_{d_1^\lambda} \xi_2 + \xi_3.$$

**Proof**: The proof is easily obtained through the characterization of $\succeq_{d_1^\lambda}$ for the case of trapezoidal fuzzy variables and the Extension Principle of Zadeh [20]. □

## 3.2 Optiminisn Dominance: Definition and Properties

Let us illustrate our idea with the particular case of two investors with different optimism levels $\alpha, \beta \in [0; 1]$. Let us set $\Pi = \{\alpha, \beta\}$ and propose the following dominance for two fuzzy variables $\xi_1$, $\xi_2$ whose respective distribution functions $\phi_\lambda^1$ and $\phi_\lambda^2$ with respect to $M_\lambda$-measure:

$$\xi_1 \geq_{D_1} \xi_2 \Rightarrow \exists \lambda \in \Pi, \forall \varepsilon \in \Pi, \phi_\lambda^1 \leq \phi_\varepsilon^2. \tag{2}$$

Intuitively, (2) means that fuzzy return $\xi_1$ is preferred to fuzzy return $\xi_2$ in that sense: there is at least an optimism level or a particular distribution of $\xi_1$ under which $\xi_1$ gives less chance to obtain lower return than $\xi_2$ whatever investor's optimism level considered for $\xi_2$ (that is whatever the way $\xi_2$ is distributed).

Let us generalize (2) by the following definition.

**Definition 3.** *Let $\xi_1$ and $\xi_2$ be two fuzzy variables whose respective distribution functions are $\phi_\lambda^1$ and $\phi_\lambda^2$ with respect to $M_\lambda$-measure.*
*The optiminisn dominance is the binary relation on fuzzy variables denoted $\succeq_{D_1}$ and defined by: $\xi_1 \succeq_{D_1} \xi_2 \Rightarrow \exists\lambda \in [0;1], \forall\beta \in [0;1]$, $\phi^1_{\lambda,\xi_1} \leq \phi^2_{\beta,\xi_2}$.*

The next result establishes that $\succeq_{D_1}$ is a pre-order on $\mathcal{A}$.

**Proposition 5.** *1. Reflexivity: $\forall\xi \in \mathcal{A}$, $\xi \succeq_{D_1} \xi$;*

*2. Symmetry: $\forall(\xi,\eta) \in \mathcal{A}^2$, if $\xi \succeq_{D_1} \eta$ and $\eta \succeq_{D_1} \xi$ then $\xi \sim \eta$;*

*3. Transitivity: $\forall(\xi,\eta,\theta) \in \mathcal{A}^3$, if $\xi \succeq_{D_1} \eta$ and $\eta \succeq_{D_1} \theta$ then $\xi \succeq_{D_1} \theta$.*

**Proof**: Let us prove the three precede properties.
*i*) Reflexivity: $\xi \succeq_{D_1} \xi$.
We assume that $\phi_\lambda$ is the distribution function of $\xi$ with respect to $M_\lambda$-measure. Let us find $\lambda_0 \in [0;1]$ such that $\forall\beta \in [0;1]$, $\forall x \in \mathbb{R}$, $\phi_{\lambda_0}(x) \leq \phi_\beta(x)$.
According to Proposition 1, it suffices to take $\lambda_0 = 0$ and we get: $\phi_0(x) \leq \phi_\beta(x), \forall x \in \mathbb{R}, \forall\beta \in [0;1]$.
*ii*) Symmetry: $\xi \succeq_{D_1} \eta$ and $\eta \succeq_{D_1} \xi \Rightarrow \xi \sim \eta$, that is $\phi^1(x) = \phi^2(x), \forall x \in \mathbb{R}$, where $\phi^1$ and $\phi^2$ are respectively the distribution functions of $\xi$ and $\eta$, defined by Liu [9] and $\phi^1_\lambda, \phi^2_\lambda$ are respectively the distribution functions of $\xi, \eta$ with respect to $M_\lambda-$ measure.
Let us assume that $\xi \succeq_{D_1} \eta$ and $\eta \succeq_{D_1} \xi$.
$\xi \succeq_{D_1} \eta \Rightarrow \exists\lambda_1 \in [0;1], \forall\beta \in [0;1], \forall x \in \mathbb{R}$, $\phi^1_{\lambda_1}(x) \leq \phi^2_\beta(x) : (R_1)$,
$\eta \succeq_{D_1} \xi \Rightarrow \exists\lambda_2 \in [0;1], \forall\beta \in [0;1], \forall x \in \mathbb{R}$, $\phi^1_{\lambda_2}(x) \leq \phi^2_\beta(x) : (R_2)$.
According to $(R_1)$,by taking $\beta = \lambda_2$, we have: $\phi^1_{\lambda_1}(x) \leq \phi^2_{\lambda_2}(x) : (R_3)$; according to $(R_2)$,by taking $\beta = \lambda_1$, we have: $\phi^2_{\lambda_2}(x) \leq \phi^1_{\lambda_1}(x) : (R_4)$.
Finally, according to $(R_3)$, $(R_4)$ and Proposition 2, we have:
$\phi^1_{\lambda_1}(x) = \phi^2_{\lambda_2}(x) \Rightarrow \phi^1(x) = \phi^2(x), \forall x \in \mathbb{R}$.
*iii*) Transitivity: $\xi \succeq_{D_1} \eta$ and $\eta \succeq_{D_1} \theta \Rightarrow \xi \succeq_{D_1} \theta$.
Let us assume that $\xi \succeq_{D_1} \eta$ and $\eta \succeq_{D_1} \theta$ where $\phi^1_\lambda, \phi^2_\lambda, \phi^3_\lambda$ are respectively the distribution functions of $\xi, \eta, \theta$ with respect to $M_\lambda-$ measure.
$\xi \succeq_{D_1} \eta \Rightarrow \exists\lambda_3 \in [0;1], \forall\beta \in [0;1], \forall x \in \mathbb{R}$, $\phi^1_{\lambda_3}(x) \leq \phi^2_\beta(x) : (R_5)$,
$\eta \geq_{D_1} \theta \Rightarrow \exists\lambda_4 \in [0;1], \forall\beta \in [0;1], \forall x \in \mathbb{R}$, $\phi^2_{\lambda_4}(x) \leq \phi^3_\beta(x)$.
According to $(R_5)$, by taking $\beta = \lambda_4$, we have: $\phi^1_{\lambda_3}(x) \leq \phi^2_{\lambda_4}(x)$. Thus, $\phi^1_{\lambda_3}(x) \leq \phi^2_{\lambda_4}(x) \leq \phi^3_\beta(x), \forall\beta \in [0;1], \forall x \in \mathbb{R}$, that is, $\phi^1_{\lambda_3}(x) \leq \phi^3_\beta(x), \forall\beta \in [0;1], \forall x \in \mathbb{R}$. Therefore, $\xi \succeq_{D_1} \theta$. $\square$

Let us characterize the optiminisn dominance $\succeq_{D_1}$ by showing that a trapezoidal fuzzy variable dominates a second one if the support of the first is on right and the one of the second is on left.

**Theorem 2.** *Let* $\xi_1 = (a_1, b_1, c_1, d_1)$ *and* $\xi_2 = (a_2, b_2, c_2, d_2)$ *be two trapezoidal fuzzy variables.*

$$\xi_1 \succeq_{D_1} \xi_2 \Longleftrightarrow a_1 \geq d_2.$$

**Proof**: Let us assume that $\phi_1^\lambda$ and $\phi_2^\lambda$ are respective distribution functions of $\xi_1$ and $\xi_2$ with respect to $M_\lambda-$ measure.
$\Leftarrow$) Let us assume that $a_1 \geq d_2$.
According to (1), it is obvious that:
$\forall\lambda \in [0;1], \forall\beta \in [0;1], \forall x \in \mathbb{R}$, $\phi_1^\lambda(x) \leq \phi_2^\beta(x)$.
Thus, $\xi_1 \geq_{D_1} \xi_2$.
$\Rightarrow$) Let us assume that $a_1 < d_2$. Let be $\lambda \in [0;1]$ and let us find $\gamma \in [0;1]$, $x_0 \in \mathbb{R}$ such that:

$$\phi_1^\lambda(x_0) > \phi_2^\gamma(x_0). \tag{3}$$

Let $x_0 \in [a_1; d_2]$ such that $x_0 \leq b_1$ and $x_0 \geq c_2$. According to (1) and (3), we have:

$$\lambda\frac{x_0 - a_1}{b_1 - a_1} > 1 + (\gamma - 1)\frac{x_0 - d_2}{c_2 - d_2}. \tag{4}$$

Let us set $\theta = \frac{d_2 - a_1}{b_1 - a_1 + d_2 - c_2}$ where $\theta$ is the solution of the equation: $\frac{x - a_1}{b_1 - a_1} = \frac{x - d_2}{c_2 - d_2}$.
We have: (4) $\Rightarrow \lambda\theta > 1 + (\gamma - 1)\theta$, that is, $\gamma < \frac{\lambda\theta - 1}{\theta} + 1$ and we set: $\varepsilon = \frac{\lambda\theta - 1}{\theta} + 1$.
i) As $\begin{cases} d_2 > a_1 \\ b_1 > a_1 \\ d_2 > c_2 \end{cases}$, it is obvious that $\theta > 0$ and thus $\varepsilon$ exits.
ii) We also have: $\theta - 1 = \frac{c_2 - b_1}{b_1 - a_1 + d_2 - c_2}$, that leads to, $\theta < 1 \Longleftrightarrow c_2 < b_1$.
Therefore, two cases have to be considered:
- If $c_2 < b_1$
Then, $\theta < 1 \Rightarrow \varepsilon < 1$ and it suffices to take $\gamma \in [0; \varepsilon[$.
- Else
We have: $\theta \geq 1 \Rightarrow \varepsilon \geq 0$.
It suffices to take $\gamma \in [0; \varepsilon \wedge 1[$. $\square$

The following result establishes the compatibility of $\succeq_{D_1}$ with respect to the sum on $\mathcal{A}$.

**Proposition 6.** *Let* $\xi_1 = (a_1, b_1, c_1, d_1)$*,* $\xi_2 = (a_2, b_2, c_2, d_2)$ *and* $\xi_3 = (a_3, b_3, c_3, d_3)$ *be three trapezoidal fuzzy variables and* $\lambda \in [0, 1]$*.*

$$\xi_1 \succeq_{D_1} \xi_2 \Rightarrow \xi_1 + \xi_3 \succeq_{D_1} \xi_2 + \xi_3.$$

**Proof**: The proof is easily obtained through the characterization of $\succeq_{D_1}$ for the case of trapezoidal fuzzy variables and the Extension Principle of Zadeh [20]. $\square$

The next result establishes that the optiminisn dominance $\succeq_{D_1}$ is strongest than the first order dominance $\succeq_{d_1^\lambda}$ in the set of trapezoidal fuzzy variables.

**Proposition 7.** *Let $\lambda \in [0;1]$ and $\xi_1, \xi_2$ be two trapezoidal fuzzy variables whose respective distribution functions are $\phi_\lambda^1$ and $\phi_\lambda^2$ with respect to $M_\lambda$-measure. Then $\xi_1 \succeq_{D_1} \xi_2 \Rightarrow \xi_1 \succeq_{d_1^\lambda} \xi_2$.*

**Proof**: The proof is easily obtained through characterization of dominances $\succeq_{D_1}$ and $\succeq_{d_1^\lambda}$ for the case of trapezoidal fuzzy variables. $\square$

Let us end this section by establishing that dominance $\succeq_{D_1}$ is a complete relation on the subset of trapezoidal fuzzy variables with disjoint supports.

**Proposition 8.** *Let $\mathcal{A}'$ be a subset of $\mathcal{A}$ which is the set of trapezoidal fuzzy variables with disjoint supports.*
*The optiminisn dominance $\succeq_{D_1}$ is complete on $\mathcal{A}'$.*

**Proof**: Let $\xi = (a, b, c, d)$ and $\xi' = (a', b', c', d')$ be two elements of $\mathcal{A}'$. As $Supp(\xi) \cap Supp(\xi') = \emptyset$, we necessarily have: $a > d'$ or $a' > d$. According to Theorem 2, we have: $\xi \succeq_{D_1} \xi'$ or $\xi' \succeq_{D_1} \xi$. $\square$

In the next Section, we use the two proposed dominances to determine the set of best portfolios of assets with fuzzy returns and we implement the obtained result on a given set of triangular fuzzy variables proposed by Huang [5].

# 4 Application of dominances for Portfolio Selection with fuzzy retun

Let us consider the family $A = (\xi)_{1 \leq i \leq n}$ of $n$ assets where returns are described by trapezoidal fuzzy numbers. A portfolio return $\xi$ associated with $A$ is a convex linear combination of the $n$ assets returns defined by $\xi = \sum_{i=1}^n x_i \xi_i$ where $x_i$ represents the proportion of capital invested in asset $i$. The set of portfolios associated with $A$ is $P = \{\xi = \sum_{i=1}^n x_i \xi_i, x_i \in [0,1], \sum_{i=1}^n x_i = 1$ and $\xi_i \in A\}$.
A main question is to determine non dominated portfolios by means of a dominance relation. Based on Game Theory terminology, we will determine the core $\mathcal{C}_R(P)$ of $(P, R)$ where $R$ is a dominance on $P$. In particular, we introduce the core of the dominances $\succeq_{D_1}$ and $\succeq_{d_1^\lambda}$ as follows:

$$\mathcal{C}_{\succeq_{d_1^\lambda}}(P) = \{\xi \in P, \forall \eta \in P \setminus \{\xi\}, \eta \not\succeq_{d_1^\lambda} \xi\} \tag{5}$$

and

$$\mathcal{C}_{\succeq_{D_1}}(P) = \{\xi \in P, \forall \eta \in P \setminus \{\xi\}, \eta \not\succeq_{D_1} \xi\}. \tag{6}$$

The following result gives a link between the two core.

**Proposition 9.** *Let* $A = (\xi)_{1\leq i\leq n}$ *be a family of* $n$ *assets where returns are described by trapezoidal fuzzy numbers and* $P$ *the set of portfolios associated with* $A$*. We have:*

$$\mathcal{C}_{\succeq_{d_1^\lambda}}(P) \subseteq \mathcal{C}_{\succeq_{D_1}}(P).$$

**Proof**: Let $\xi = (a, b, c, d)$ be a trapezoidal fuzzy variable such that $\xi \in \mathcal{C}_{\succeq_{d_1^\lambda}}(P)$. Let us prove that $\xi \in \mathcal{C}_{\succeq_{D_1}}(P)$. Let $\eta = (a', b', c', d') \in P\backslash\{\xi\}$.
$\xi \in \mathcal{C}_{\succeq_{d_1^\lambda}}(P) \Rightarrow a > a'$ or $b > b'$ or $c > c'$ or $d > d'$.
Without loss of generality, let us assume that $b > b'$.
If $a' \geq d$ then we obtain: $b' \geq a' \geq d \geq b$, that is $b' \geq b$, which contradicts the assumption. So, we conclude that $a' < d$. More precisely, $\eta \not\succeq_{D_1} \xi$ and $\xi \in \mathcal{C}_{\succeq_{D_1}}(P)$. □

Notice that if $\lambda = \frac{1}{2}$, then Proposition 9 implies: $\mathcal{C}_{\succeq_{d_1^{\frac{1}{2}}}}(P) \subset \mathcal{C}_{\succeq_{D_1}}(P)$, that is, the new set $\mathcal{C}_{\succeq_{D_1}}(P)$ of non dominated portfolios contains non dominated portfolios obtained with the first order dominance associated with credibility measure obtained by Tassak et al. [17].

The following result characterizes some non dominated portfolios in $\mathcal{C}_{\succeq_{D_1}}(P)$.

**Proposition 10.** *Let* $A = (\xi = (a_i, b_i, c_i, d_i))_{1\leq i\leq n}$ *be a family of* $n$ *assets where returns are described by trapezoidal fuzzy numbers and* $P$ *the set of portfolios associated with* $A$*. Let us set* $\Gamma_1 = \{\xi = (a, b, c, d)/a = \max_{1\leq i\leq n} \sum_{i=1}^n x_i a_i\}$ *and* $\Gamma_2 = \{\xi = (a, b, c, d)/d = \max_{1\leq i\leq n} \sum_{i=1}^n x_i d_i\}$*.*

1. $\Gamma_1 \subset \mathcal{C}_{\succeq_{D_1}}(P)$.
2. $\Gamma_2 \subset \mathcal{C}_{\succeq_{D_1}}(P)$.

**Proof**: The proof is easily obtained by the characterization of $\succeq_{D_1}$ according to Theorem 2. □

By considering the family $A = (\xi = (a_i, b_i, c_i, d_i))_{1\leq i\leq n}$ of $n$ assets where returns are described by trapezoidal fuzzy numbers and $P$ the set of portfolios associated with $A$, Proposition 10 leads us to propose two models for the determination in $\Gamma_1$ and $\Gamma_2$ of non dominated portfolios of with respect to the dominance $\succeq_{D_1}$ as follows:

$$\begin{cases} \max a_1 x_1 + ... + a_n x_n \\ \sum_{i=1}^n x_i = 1 \\ x_i \in [0, 1], \forall i \in \{1, \ldots, n\} \end{cases}. \tag{7}$$

and

$$\left\{ \begin{array}{l} \max d_1x_1 + ... + d_nx_n \\ \sum_{i=1}^{n} x_i = 1 \\ x_i \in [0,1], \forall i \in \{1,\dots,n\} \end{array} \right. . \quad (8)$$

In what follows, we implement the two previous optimization models to determine best portfolios in $\Gamma_1$ and $\Gamma_2$ of the set of seven assets described by seven triangular fuzzy variables in the following table proposed by Huang [5].

| Security i | Fuzzy return | Security i | Fuzzy return |
|---|---|---|---|
| 1 | $\xi_1 = (-0.3, 1.8, 2.3)$ | 5 | $\xi_5 = (-0.7, 2.4, 2.7)$ |
| 2 | $\xi_2 = (-0.4, 2.0, 2.2)$ | 6 | $\xi_6 = (-0.8, 2.5, 3.0)$ |
| 3 | $\xi_3 = (-0.5, 1.9, 2.7)$ | 7 | $\xi_7 = (-0.6, 1.8, 3.0)$ |
| 4 | $\xi_4 = (-0.6, 2.2, 2.8)$ | | |

Table 1: Fuzzy returns of 7 securities (units per stock).

Implementation of models (7) and (8) made with Matlab provides respectively two non dominated portfolios with respect to dominance $\succeq_{D_1}$ defined by: $\xi = \xi_1$ and $\eta = 0.4214\xi_6 + 0.5786\xi_7$.
We observe that: $\xi$ is obtained by investing all the capital in asset 1 which expresses the "best loss" ($a = -0.3$) whereas $\eta$ is obtained by investing the capital only in asset 6 and asset 7 which express the "highest gain" ($c = 3.0$). So, best portfolios $\xi$ and $\eta$ are respectively choice of extreme risk averse and extreme risk taker investors.

The following table presents characteristics such as mean, variance and semi-variance, kurtosis and skewness of our optimal portfolios and those obtained by previous authors (Huang [5], Li et al. [6], Sadefo et al. [14] and Tassak et al. [17]).

| | Mean | Variance | Kurtosis | Skewness |
|---|---|---|---|---|
| Huang [5] | 1.6 | 0.7235 | 1.7972 | -0.7543 |
| Li et al. [6] | 1.6 | 0.7019 | 1.7291 | -0.6823 |
| Sadefo et al. [14] | 1.6 | 0.7018 | 1.7290 | -0.6823 |
| Tassak et al. [17] | 1.5605 | 0.6973 | 1.7033 | -0.6666 |
| Model (7) | 1.4 | 0.4434 | 0.6922 | -0.3380 |
| Model (8) | 1.5975 | 0.9863 | 2.4179 | -0.7950 |

Table 2 : Characteristics of optimal portfolios obtained from our two models and previous models

From Table 2, optimal portfolios' characteristics present two different investors' behaviors: averse risk investors and risky investors respectively from

model (7) and model (8). The first category prefers best risk values described by smallest variance, smallest kurtosis and highest skewness, even though the obtained benefit is lower than the one obtained by previous authors ([5], [6],[14], [17]). Thus, since risk values obtained by model (7) are better than those obtained by the afore-mentioned authors, this portfolio is the most convenient for risk averse investors. The second category prefers high benefit (1.5975) even though risk values are very uncomfortable.
Except the fact that model (7) and model (8) proposed optimal portfolios for two different types of investors as the recent ones obtained respectively by Sadefo et al. [14] and Tassak et al. [17], they focus in particular on extreme cases: investors extremely risk averse (model (7)) and investors extremely risky (model (8)). Moreover, those results justified that, as future portfolio's return is determined through capital's proportions investment on assets, future assets' returns comparisons (through dominance) brings more efficiency on the way of sharing a capital on different portfolio's assets.

## 5 Concluding remarks

In this paper, two new dominance relations based on the parametric fuzzy measure were introduced, characterized following two main assumptions and their properties were studied. The obtained results on these new dominances complement previous contributions on various studies made on ranking fuzzy variables, in particular on trapezoidal fuzzy variables, by many scholars ([1], [4], [19], [12]). Furthermore, we implement the optiminisn dominance (the stronger one) on the set of seven assets proposed by Huang [10] to enlarge the set of best portfolios obtained by Tassak et al. [17] through the first order dominance. Those obtained portfolios complement previous and recent studies ([5], [6], [14], [17]) on portfolio selection with fuzzy return by means of the credibility fuzzy measure (when parameter is $\frac{1}{2}$).
In the next research, we intend to introduce and study a new dominance relations which take into account investors optimism's level to compare two fuzzy variables and based on partial moment of those fuzzy variables defined through the generalized fuzzy measure.

## Appendix

**Proof of Proposition 2**:
Let $\xi_1 = (a_1, b_1, c_1, d_1)$ and $\xi_2 = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy variables whose respective distribution functions $\phi^1_\lambda$ and $\phi^2_\lambda$ with respect to $M_\lambda-$ measure.
Let us assume that there exist $\lambda_1, \lambda_2 \in [0;1]$ such that

$$\phi^1_{\lambda_1}(x) = \phi^2_{\lambda_2}(x), \forall x \in \mathbb{R} \tag{9}$$

and let us prove that:

$$\phi^1(x) = \phi^2(x), \forall x \in \mathbb{R}.$$

1) <u>First step</u>

Let us prove that $\begin{cases} a_1 = a_2, \\ b_1 = b_2, \\ c_1 = c_2, \\ d_1 = d_2 \end{cases}$

i)If $a_1 \neq a_2$ (without loss of generality, we assume that $a_1 < a_2$).

For x $\in ]a_1; a_2[$, according to (1) we have:

$\phi^1_{\lambda_1}(x) \neq 0$ and $\phi^2_{\lambda_2}(x) = 0$, that contradicts (9).

ii) If $b_1 \neq b_2$ (without loss of generality, we assume that $b_2 < b_1$).
For $x_1, x_2 \in ]b_2; b_1[$ such that $x_1 \neq x_2$ and $x_1 < c_1$ , $x_2 < c_2$, according to (1) we have:
$\phi^1_{\lambda_1}(x_1) = 0$ or $\phi^2_{\lambda_2}(x_2) = 0$ or $\phi^1_{\lambda_1}(x_1) = \phi^2_{\lambda_2}(x_2) = \lambda_2$. The first two cases are obvious and we focus only on the third case:
$\phi^1_{\lambda_1}(x_1) = \phi^2_{\lambda_2}(x_2) = \lambda_2 \Longrightarrow \frac{\lambda_1(x_1-a_1)}{b_1-a_1} = \frac{\lambda_1(x_2-a_1)}{b_1-a_1} = \lambda_2$, according to (1) and (9). We have:
$(\frac{\lambda_1(x_1-a_1)}{b_1-a_1} = \frac{\lambda_1(x_2-a_1)}{b_1-a_1} = \lambda_2) \Rightarrow x_1 = x_2$, which contradicts $x_1 \neq x_2$.

iii) If $c_1 \neq c_2$ (without loss of generality, we assume that $c_2 < c_1$).
For $x_1, x_2 \in ]c_2; c_1[$ such that $x_1 \neq x_2$ and $x_1 > b_1$ , $x_2 > b_2$, according to (1) we have:
$\phi^1_{\lambda_1}(x_1) = 1$ or $\phi^2_{\lambda_2}(x_2) = 1$ or $\phi^1_{\lambda_1}(x_1) = \phi^2_{\lambda_2}(x_2) = \lambda_1$. The first two cases are obvious and we focus only on the third case:
$\phi^1_{\lambda_1}(x_1) = \phi^2_{\lambda_2}(x_2) = \lambda_1 \Longrightarrow \frac{\lambda_2(d_2-x_1)+x_1-c_2}{d_2-c_2} = \frac{\lambda_2(d_2-x_2)+x_2-c_2}{d_2-c_2} = \lambda_1$ according to (1) and (9). We have:
$(\frac{\lambda_2(d_2-x_1)+x_1-c_2}{d_2-c_2} = \frac{\lambda_2(d_2-x_2)+x_2-c_2}{d_2-c_2} = \lambda_1) \Longrightarrow x_1 = x_2$, that contradicts $x_1 \neq x_2$.

iv) If $d_1 \neq d_2$ (without loss of generality, we assume that $d_1 < d_2$).
For $x \in ]d_1; d_2[$, according to (1) we have: $\phi^1_{\lambda_1}(x_1) = 1$ and $\phi^2_{\lambda_2}(x) < 1$, that contradicts (9).
Finally, we have $a_1 = a_2, b_1 = b_2, c_1 = c_2, d_1 = d_2$.

2) <u>Second step</u>

Let us prove that $\lambda_1 = \lambda_2$.
According to the first step, we have $[b_2, c_2] = [b_1, c_1]$. Moreover, $x \in [b_1, c_1] \Longrightarrow \phi^1_{\lambda_1}(x_1) = \lambda_1$ and $x \in [b_2, c_2] \Longrightarrow \phi^2_{\lambda_2}(x) = \lambda_2$.
Thus: $(x \in [b_1, c_1] \Longleftrightarrow x \in [b_2, c_2])$ and $\phi^1_{\lambda_1}(x_1) = \phi^2_{\lambda_2}(x)$, that is, $\lambda_1 = \lambda_2$.

3) Third step

Let us prove that $\phi^1(x) = \phi^2(x), \forall x \in \mathbb{R}$.
By the fact that,
$\phi^1_{\lambda_1}(x) = \phi^2_{\lambda_2}(x), \forall x \in \mathbb{R}, \lambda_1 = \lambda_2$ and

$$\begin{cases} a_1 = a_2, \\ b_1 = b_2, \\ c_1 = c_2, \\ d_1 = d_2 \end{cases}$$

we can conclude according to (1) that:
$\phi^1_{\lambda_1}(x_1) = \phi^2_{\lambda}(x), \forall x \in \mathbb{R}, \forall \lambda \in [0,1]$, that means in the particular case where $\lambda = \frac{1}{2}$ (Liu [9]), $\phi^1(x) = \phi^2(x), \forall x \in \mathbb{R}$. $\square$

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